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Diffusion coefficients as function of Kubo number in random fields

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Abstract. Particle standard diffusion $D(K)$ in 2D homogeneous, isotropic, stationary, random velocity fields is studied numerically as a function of the dimensionless Kubo number K defined as the ratio between the correlation time of the velocity field and its sweeping time. The three cases corresponding to the energy spectra proportional, in the inertial range, respectively to k^{-3}, $k^{-\frac{5}{3}}$ and k^2 are considered and two different power law regimes are found for large but finite K.

1. Introduction

In the passive transport of mass or heat the coupling between advection and molecular diffusivity is a difficult problem of considerable practical and theoretical interest in a great variety of different fields, such as geophysics, chemical engineering and disordered media [1,2].

Taking into account the molecular diffusion, the Lagrangian motion of a test particle is described by the Langevin equation

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t) + \boldsymbol{\xi} \tag{1}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the Eulerian velocity field at the position $\boldsymbol{x}$ and the time t, and ξ is a Gaussian white noise with zero mean and correlation function

$$\langle \xi_i(t)\xi_j(t')\rangle = 2D_0\delta_{ij}\delta(t - t') \tag{2}$$

the coefficient D_0 being the molecular diffusivity. If Θ is the density of tracers, the Fokker–Planck equation associated to (1) is

$$\partial_t\Theta + \boldsymbol{\partial}(\boldsymbol{v}\Theta) = D_0\partial^2\Theta. \tag{3}$$

For times much larger than the typical time of $\boldsymbol{v}$, the large-scale density field $\langle\Theta\rangle$ (i.e. the field Θ averaged over a volume of linear dimension much larger than the typical length of the velocity field $\boldsymbol{v}$) obeys a standard diffusion Fick equation:

$$\partial_t\langle\Theta\rangle = D^E_{ij}\partial^2_{x_i x_j}\langle\Theta\rangle \qquad i, j = 1, \ldots, d. \tag{4}$$

All the (often nontrivial) effects due to the presence of the velocity field are in the eddy diffusion coefficient D^E_{ij}. Of course if the equation (4) holds then one has $\langle(x(t) - x(0))^2\rangle \simeq 2D^E_{11}t$ and we speak of *standard diffusion*. In practice at large time the test particle behaves as a Brownian particle. There also exists the possibility of anomalous diffusion, i.e. $\langle(x(t) - x(0))^2\rangle \simeq t^{2\nu}$

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with $\nu > \frac{1}{2}$, see e.g. [3]. Here we consider only fields for which standard diffusion surely occurs [4, 5].

The multiscale technique allows us to compute [6, 7, 10] the eddy diffusivity tensor by solving the *auxiliary equation*

$$[\partial_t + \boldsymbol{\partial}(\boldsymbol{v}\cdot) - D_0\partial^2]\boldsymbol{\chi} = -\boldsymbol{v}. \tag{5}$$

Actually, the effective eddy diffusivity tensor D_{ij}^E is given by

$$D_{ij}^E = D_0\delta_{ij} - \tfrac{1}{2}(\langle v_i\chi_j\rangle + \langle v_j\chi_i\rangle) \tag{6}$$

where the vector field $\boldsymbol{\chi}$, the *auxiliary field*, obeys (5). Numerical methods are generally needed to solve it, but there are a few cases where one can obtain the auxiliary field analytically [8, 10–13].

Of course, one can also compute the diffusion coefficients with a direct numerical simulation of an ensemble of particles evolving according to equation (1). Nevertheless, a numerical study of equation (5), and then the computation of D_{ij}^E using (6), has some advantages. Since in the multiscale technique one already works in the asymptotic regime it is possible to avoid unpleasant long crossover regimes in the diffusion process which can give difficulties in the computation of the diffusion coefficients. In this paper we consider the passive transport in a 2D incompressible, statistically stationary, homogeneous and isotropic random velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, specified by the correlation tensor

$$\langle v_i(\boldsymbol{x}, t)v_j(\boldsymbol{x}', t')\rangle = C_{ij}(\boldsymbol{x}, \boldsymbol{x}', t, t') \tag{7}$$

and related to the energy spectrum $E(k)$ by the equation

$$E = \int_0^\infty \mathrm{d}k\, E(k) = \sum_i \langle v_i v_i\rangle(0, 0) = \sum_i C_{ii}(0, 0) = v_0^2 \tag{8}$$

as specified in appendix A. We analyse the behaviour of the diffusion coefficients in random fields mimicking fully developed turbulence as a function of the dimensionless Kubo number K [14], also known as the Strouhal number [15],

$$K = v_0\tau_c/\lambda_c \tag{9}$$

which is the ratio between the correlation time τ_c of the velocity field and its *sweeping time*, defined as $T_{sw} = \lambda_c/v_0$ where λ_c is the characteristic length of the velocity field. Taking $T_{sw} \sim 1$, the Kubo number is a dimensionless measure of τ_c. We will analyse the 2D case essentially for two reasons: the first one is that in this case there already exists an analytical prediction for the behaviour of $D(K)$ as a function of very large but finite K [16] and the second one is that the 2D problem is more convenient from a numerical point of view. However, we do not expect a strong dependence of our analysis on the space dimension.

In section 2 we briefly review some previous works on rather simplified models of random fields. Section 3 is devoted to the introduction and the diffusive properties study of more complicated random fields. We shall see that the asymptotic behaviour, i.e. for $K \gg 1$, is not universal but depends on the details of the fields. Section 4 contains our conclusions.

2. The problem

In the case of a 2D, homogeneous, isotropic, stationary delta-correlated in time random field, i.e.

$$C_{ij}(\boldsymbol{x}, \boldsymbol{x}', t, t') = R_{ij}(\boldsymbol{r})T(|t - t'|) \tag{10}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ and

$$T(|t-t'|) = 2\tau_0\delta(t-t') \tag{11}$$

the eddy diffusivity tensor can be calculated using the Taylor formula [17] obtaining

$$D = D_{11}^E = D_{22}^E = D_0 + \tfrac{1}{2}v_0^2\tau_0 \tag{12}$$

where v_0^2 is done by equation (8).

For non delta-correlated in time velocity fields, i.e. with $T(|t-t'|)$ defined as

$$T(|t-t'|) = \frac{2\tau_0}{\tau_c}\mathrm{e}^{-\frac{|t-t'|}{\tau_c}} \tag{13}$$

a useful dimensionless parameter is the Kubo number introduced in the previous section. An interesting question is how the diffusive properties of the system changes as K varies.

This problem has been studied in the case $D_0 = 0$ by Gruzinov *et al* [19], Isichenko [16] and by Ottaviani [18]. The random velocity field, studied in [18], corresponds to

$$T(|t-t'|) = A^2\mathrm{e}^{-\frac{|t-t'|}{\tau_c}} \qquad \text{and} \qquad E(k) = \frac{\lambda_c^4 u_0^2}{2}k^3\mathrm{e}^{-\frac{\lambda_c^2}{2}k^2} \tag{14}$$

where the spectrum $E(k)$ peaks at $k = \frac{\sqrt{3}}{\lambda_c}$. This spectrum has also been studied by Kraichnan in [20], comparing the particle diffusion calculated in two and three dimensions by computer simulation with that obtained by the direct-interaction approximation (DIA).

For $K \ll 1$, analytical approximations of $D(K)$ are available [10]; at the first order in $\tau_c \sim K$ one has

$$D^E = \frac{D_{11}^E + D_{22}^E}{2} = \frac{1}{2}\int \mathrm{d}k\,E(k)\int_0^\infty \mathrm{d}t\,T(t) \simeq \frac{v_0^2}{2}A^2\tau_c \sim K. \tag{15}$$

By contrast, for infinite K one can predict $D(K=\infty) = 0$ for a generic two-dimensional incompressible *frozen* velocity field. This is because in such a field the test particles move along the streamlines, which are closed with probability 1 in the generic case. In the case of open streamlines like a steady shear flow due to the ballistic motion, superdiffusion takes place.

As discussed in [18] the interesting problem is the case $K \gg 1$ but not infinite. Actually, in this regime, an estimate of $D(K)$ with simple argument is not trivial. The difficulty comes from the fact that whereas most of the particles are constrained to almost closed trajectories of size $a = \mathrm{O}(\lambda_c)$ for long times of order $\tau_c \sim K$, a small fraction is allowed much longer excursions with much shorter correlation times. Over a long time, any given particle would experience long periods of small displacements (effective trapping) and short periods of long excursions, where most of the contribution to the diffusivity comes from.

In [16, 21] the authors introduce the asymptotic exponent $0 < \alpha < 1$ defined as

$$D(K) \sim K^{-\alpha} \qquad K \gg 1. \tag{16}$$

An estimate of α is done in [19] and [16] following a percolation theory analysis of effective diffusion in a two-dimensional random, incompressible, time-dependent flow and the suggested value is $\frac{3}{10}$.

The numerical simulations in [18] give the correct linear behaviour of $D(K)$ for small K, but in the opposite regime the found value of α is close but different ($\alpha = 0.2 \pm 0.04$) from the one predicted by the percolation theory. The linear behaviour takes place from $K \simeq 10^{-3}$ to $K \simeq 1$, then there is a transition region between $K \simeq 1$ and $K \simeq 10^2$ and finally the power law occurs in the range $K \in [10^2, 10^4]$. The crossover around $K = 1$ can be physically explained as follows. When $K < 1$, before having the time to travel across λ_c, the tracer particle experiences many different changes of the velocity field (because $\tau_c < T_{sw}$); therefore the Lagrangian correlation time τ_L (the correlation time of the particle) is of the order of τ_c. Because we are interested in the tracer's behaviour over distances much larger than λ_c, taking as elementary time step τ_c, the motion of the particle can be thought of as a Brownian motion corresponding to a linear diffusivity $D \sim \tau_c$.

3. Random advective flow with non-separable correlation function: numerical results

The random field discussed in the previous section has a separable correlation function which implies scale-independent correlation times and thus a rough description of turbulence. Actually, in turbulent flows many different scales and thus many differents times are involved.

This is why we introduce random fields with non-separable correlation functions and with different characteristic times $\tau(k)$ corresponding to different scales $1/k$.

Let us consider a random field periodic in space, with period L in both directions and with correlation function $C_{ij}(\boldsymbol{x}, \boldsymbol{x}', t, t')$, the Fourier transform of which is

$$\hat{C}_{ij}(\boldsymbol{k}, \boldsymbol{k}', t, t') = (2\pi)^2 \delta(\boldsymbol{k} + \boldsymbol{k}')\hat{R}_{ij}(\boldsymbol{k})\hat{T}(\boldsymbol{k}, |t - t'|). \tag{17}$$

The function $\hat{R}_{ij}(\boldsymbol{k})$ and $\hat{T}(\boldsymbol{k}, |t - t'|)$ are defined as

$$\hat{R}_{ij}(\boldsymbol{k}) = (\delta_{ij}k^2 - k_i k_j) f(k) \qquad \text{and} \qquad \hat{T}(\boldsymbol{k}, |t - t'|) = \mathrm{e}^{-\frac{|t-t'|}{\tau(k)}}. \tag{18}$$

$\tau(k)$ is the characteristic time of the wavevector $\boldsymbol{k}$, the function $f(k)$ is related to the field energy spectrum $E(k)$ (see appendix A) by the formula

$$E(k) = \frac{1}{2\pi} k^3 f(k) \tag{19}$$

with $\int \mathrm{d}k E(k) = \sum_i C_{ii}(0, 0) = v_0^2$, and the characteristic length λ_c of the system has been evaluated as

$$\lambda_c^2 = \frac{\int \mathrm{d}k E(k) k^{-2}}{\int \mathrm{d}k E(k)} = \frac{\int \mathrm{d}k E(k) k^{-2}}{v_0^2}. \tag{20}$$

We have considered the following three spectra.

Case 1. The spectrum $E(k)$ is defined as

$$E_1(k) = \begin{cases} A_1 k^3 & k \in [0, k_m^I] \\ B_1 k^{-3} & k \in [k_m^I, k_M^I] \\ C_1 \mathrm{e}^{-k} & k > k_M^I. \end{cases} \tag{21}$$

Case 2. The spectrum $E(k)$ is defined as

$$E_2(k) = \begin{cases} A_2 k^3 & k \in [0, k_m^I] \\ B_2 k^{-\frac{5}{3}} & k \in [k_m^I, k_M^I] \\ C_2 \mathrm{e}^{-k} & k > k_M^I. \end{cases} \tag{22}$$

Case 3. The spectrum $E(k)$ is defined as

$$E_3(k) = \begin{cases} A_3 k^3 & k \in [0, k_m^I] \\ B_3 k^{2} & k \in [k_m^I, k_M^I] \\ C_3 \mathrm{e}^{-k} & k > k_M^I. \end{cases} \tag{23}$$

The first spectrum corresponds to the Kraichnan–Batchelor model for the 2D, stationary, homogeneous and isotropic turbulence [22, 23], say to the bidimensional turbulence in the *direct enstrophy cascade regime*; the second one corresponds to the bidimensional turbulence in the *inverse energy cascade regime* [23] which is equal to the spectrum predicted by the Kolmogorov theory for the 3D, stationary, homogeneous and isotropic turbulence. Moreover, the first two spectra correspond to *local interactions* between the scales in the inertial range of the corresponding velocity fields [24] while the third one does not.

In all the cases, following the definition of *eddy turnover time* associated with the scale $l \sim 1/k$ in the turbulence phenomenology [24], the correlation time function $\tau(k)$ has been chosen equal to $1/(k\hat{v}(k)) \simeq k^{-\frac{3}{2}} E(k)^{-\frac{1}{2}}$, namely:

Case 1. The correlation time function is

$$\tau_1(k) = c_\tau \cdot \begin{cases} D_1 k^{-3} & k \in (0, k_m^I] \\ 1 & k \in [k_m^I, k_M^I] \\ G_1 k^{-\frac{3}{2}} \mathrm{e}^{\frac{k}{2}} & k > k_M^I. \end{cases} \tag{24}$$

Case 2. The correlation time function is

$$\tau_2(k) = c_\tau \cdot \begin{cases} D_2 k^{-3} & k \in (0, k_m^I] \\ k^{-\frac{2}{3}} & k \in [k_m^I, k_M^I] \\ G_2 k^{-\frac{3}{2}} \mathrm{e}^{\frac{k}{2}} & k > k_M^I. \end{cases} \tag{25}$$

Case 3. The correlation time function is

$$\tau_3(k) = c_\tau \cdot \begin{cases} D_3 k^{-3} & k \in (0, k_m^I] \\ k^{-\frac{5}{2}} & k \in [k_m^I, k_M^I] \\ G_3 k^{-\frac{3}{2}} \mathrm{e}^{\frac{k}{2}} & k > k_M^I. \end{cases} \tag{26}$$

The constants A_i, C_i and D_i, G_i are chosen to ensure the continuity of $E(k)$ and $\tau(k)$ in k_m^I and k_M^I. B_i are determined by the condition $\int \mathrm{d}k E(k) = v_0^2$. The expression of the constants as functions of k_m^I and k_M^I is reported in appendix B. The characteristic correlation time τ_c of the random fields can be estimated as $\tau(k_{\max})$ where $k_{\max}$ is the wavenumber corresponding to the maximum of the energy spectrum. Thus in the first two cases $\tau_c = \tau(k_m^I)$, while in the third one $\tau_c = \tau(k_M^I)$. For (24)–(26) the characteristic time corresponds to the minima of the correlation time function $\tau(k)$. The sweeping time T_{sw} is fixed once the characteristic lengths $\lambda_c^{(i)}$ are calculated by (20). Therefore the Kubo number, $K = v_0\tau_c/\lambda_c$, is changed varying c_τ in (24)–(26).

For $K \ll 1$ using (15) one has the linear behaviour

$$D^E = \frac{D_{11}^E + D_{22}^E}{2} \simeq D_0 + \tfrac{1}{2} \int \mathrm{d}k E(k) \int_0^\infty \mathrm{d}t T(k,t)$$
$$= D_0 + \tfrac{1}{2} \int \mathrm{d}k E(k)\tau(k) = D_0 + \tfrac{1}{2} D_c c_\tau \tag{27}$$

where $D_c = \int \mathrm{d}k E(k)\tau(k)/c_\tau$. The constant c_τ is linearly proportional to the Kubo number via the definition of τ_c and therefore the linear behaviour is found. In the opposite limit, $K \gg 1$, due to the presence of the molecular diffusion and the almost *frozen* field we expect that $D(K)$ assume a value which does not depend on K but just on D_0 and the steady velocity field. In figures 1–3 the non-dimensional diffusion $(D(K) - D_0)/(v_0^2 T_{sw})$ is shown as a function of K.

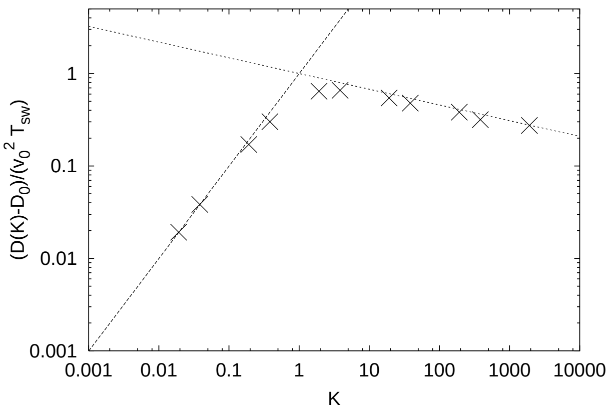

Figure 1. The dimensionless difference $(D(K) - D_0)/(v_0^2 T_{sw})$ versus K corresponding to the case (21) and (24) for $D_0/(v_0^2 T_{sw}) = 0.005\,74$. The dashed line corresponds to the predicted linear behaviour and the other corresponds to the power law behaviour $K^{-\alpha}$ with $\alpha = 0.17 \pm 0.015$.

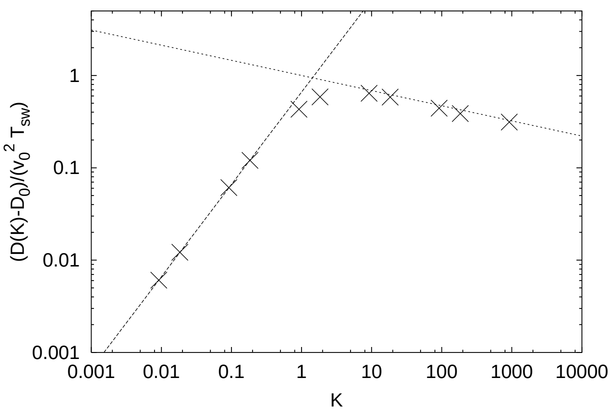

Figure 2. The dimensionless difference $(D(K) - D_0)/(v_0^2 T_{sw})$ versus K corresponding to the case (22) and (25) for $D_0/(v_0^2 T_{sw}) = 0.005\,74$. The dashed line corresponds to the predicted linear behaviour and the other corresponds to the power law behaviour $K^{-\alpha}$ with $\alpha = 0.16 \pm 0.015$.

It is evident that the predicted linear behaviour of $D(K)$ takes place for $K < 1$ while after a transition region a power law behaviour $K^{-\alpha}$ seems to occur with two different values of the exponent α: $\alpha_1 \simeq \alpha_2 \simeq 0.165$ while $\alpha_3 \simeq 0.045$.

The above results have been obtained performing direct numerical simulations of the auxiliary equation (5) by using a pseudo-spectral method [25] over the periodic box $2\pi \times 2\pi$ with a resolution 64×64, de-aliasing has been obtained by a circular truncation for $|k| \geqslant \frac{64}{3}$ [26, 27].

In order to assure the incompressibility of the bidimensional velocity field we have introduced the stream function ψ in such a way that

$$v_1 = \frac{\partial \psi}{\partial x_2} \qquad v_2 = -\frac{\partial \psi}{\partial x_1} \tag{28}$$

which corresponds in the Fourier space to

$$\hat{v}_1 = +\mathrm{i}k_2\hat{\psi} \qquad \hat{v}_2 = -\mathrm{i}k_1\hat{\psi}. \tag{29}$$

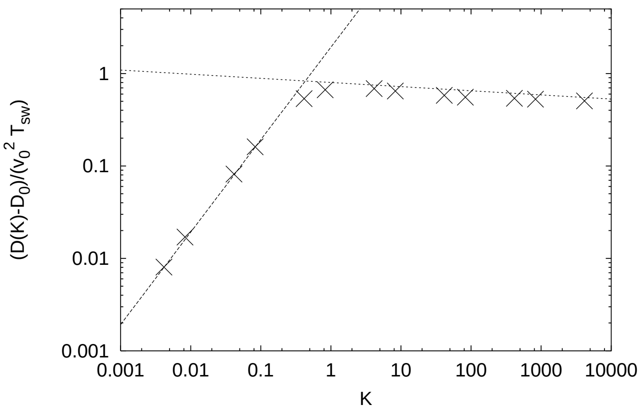

Figure 3. The dimensionless difference $(D(K) - D_0)/(v_0^2 T_{sw})$ versus K corresponding to the case (23) and (26) for $D_0/(v_0^2 T_{sw}) = 0.005\,74$. The dashed line corresponds to the predicted linear behaviour and the other corresponds to the power law behaviour $K^{-\alpha}$ with $\alpha = 0.045 \pm 0.003$.

Using (29) and (18) the velocity correlation function (17) can thus be substituted by the stream function correlation which results in

$$\langle \hat{\psi}(\boldsymbol{k}, t)\hat{\psi}(\boldsymbol{k}', t')\rangle = (2\pi)^2 \delta(\boldsymbol{k} + \boldsymbol{k}') f(k) T(k, |t - t'|). \tag{30}$$

In order to satisfy relation (30) the stream function has been generated taking

$$\hat{\psi}(\boldsymbol{k}, t + \Delta t) = a\hat{\psi}(\boldsymbol{k}, t) + b\hat{w}(\boldsymbol{k}, t) \tag{31}$$

$$a = \mathrm{e}^{-\frac{\Delta t}{\tau(k)}} \qquad b = \sqrt{1 - a^2} \tag{32}$$

where $\hat{w}(\boldsymbol{k}, t)$ is a zero average Gaussian variable such that

$$\langle \hat{w}(\boldsymbol{k}, t)\hat{w}(\boldsymbol{k}', t')\rangle = (2\pi)^2 \frac{D_0}{\tau(k)} f(k)\delta(\boldsymbol{k} + \boldsymbol{k}')\delta(t - t') \tag{33}$$

generated taking

$$\begin{cases} \mathrm{Re}\,\hat{w}(\boldsymbol{k}, t) = (2\pi) b_1(\boldsymbol{k}, t)\sqrt{D_0 f(k)/(2\tau(k))} \\ \mathrm{Im}\,\hat{w}(\boldsymbol{k}, t) = (2\pi) b_2(\boldsymbol{k}, t)\sqrt{D_0 f(k)/(2\tau(k))} \end{cases} \tag{34}$$

with b_1 and b_2 Gaussian numbers such that

$$\langle b_1(\boldsymbol{k}, t) b_1(\boldsymbol{k}', t')\rangle = \delta(\boldsymbol{k} + \boldsymbol{k}')\delta(t - t') \tag{35}$$

$$\langle b_2(\boldsymbol{k}, t) b_2(\boldsymbol{k}', t')\rangle = \delta(\boldsymbol{k} + \boldsymbol{k}')\delta(t - t') \tag{36}$$

$$\langle b_1(\boldsymbol{k}, t) b_2(\boldsymbol{k}', t')\rangle = 0. \tag{37}$$

In this way, due to equation (19), we are sure that at each time the assumed spectra (21)–(23) for the random velocity field are actually achieved. Figure 4 shows the spectra of the random velocity field at time $10^3 \Delta t$ for the spectrum of case 1. Figure 5 shows the temporal correlation function of the random velocity field calculated for $|t - t'| = 10\Delta t$ in case 1.

The boundaries of the inertial range have been fixed at $k_m^I = 4$ and $k_M^I = 18$ in order to maximize the inertial range ensuring a good numerical approximation of the condition $\int \mathrm{d}k E(k) = v_0^2 = 1$.

The convergence of the algortithm has been tested using steady and time-depending random shear flows parallel to the x-direction. In these cases, actually, the analytical expression

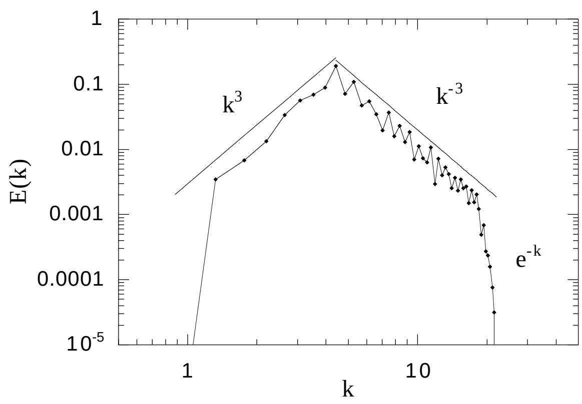

Figure 4. The spectrum $E(k)$ versus k in the case (21) and (24) at time $10^3\Delta t$ for $D_0/(v_0^2 T_{sw}) = 0.005\,74$, $K = 3.83$. The spectra have been averaged over 10^3 realizations of the random velocity field.

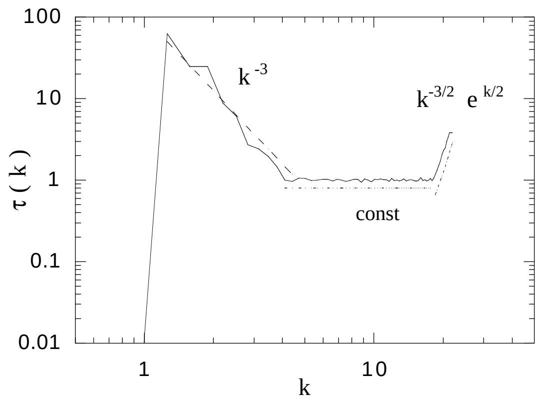

Figure 5. The correlation time function $\tau(k)$ versus k in the case (21) and (24) for $|t - t'| = 10\Delta t$ and $D_0/(v_0^2 T_{sw}) = 0.005\,74$, $K = 3.83$. The correlation time have been calculated by averaging over 10^3 realizations of the random velocity field.

of the eddy diffusivity tensor is known [4, 10] and for time-dependent velocity fields results in

$$D_{11}^E = D_0\left(1 + \frac{1}{(2\pi)^4}\int \mathrm{d}k \int \mathrm{d}\omega \frac{k^2}{\omega^2 + D_0^2 k^4}\langle|\hat{v}(k,\omega)|^2\rangle\right) \tag{38}$$

$$D_{22}^E = D_0 \qquad D_{12}^E = 0. \tag{39}$$

The calculated diffusion coefficients agree with the theoretical values (38) and (39) within errors of less than 1%.

The eddy diffusivity tensor has been calculated up to time $\sim 5 \times 10^3 \tau_c$ with a time step $\Delta t \simeq 10^{-3}\tau_c$ which, however, has to be smaller than $\Delta t_{\max} \simeq 2 \times 10^{-4}\tau_c/c_\tau$. This upper limit has been fixed in order to avoid the temporal instability of the algorithm due to the advecting term of the auxiliary equation [28]. Doubling the resolution of the numerical method, the calculated values of the eddy diffusivity tensor do not change significantly.

In figure 6 an example of temporal convergence of the calculated diffusivity D^E is shown.

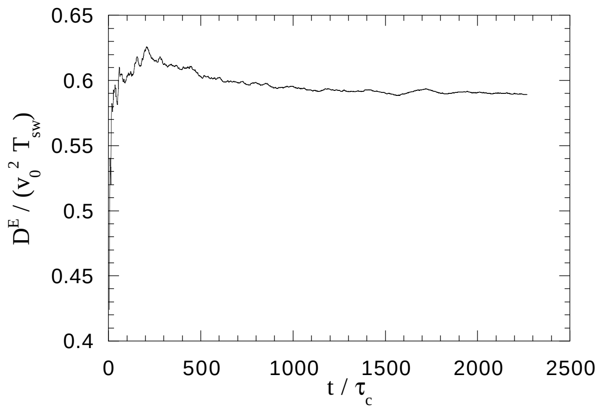

Figure 6. The dimensionless diffusivity $D^E/(v_0^2 T_{sw})$ with $D^E = (D_{11}^E + D_{22}^E)/2$ and D_{ij}^E done by equation (6) versus t/τ_c in the case (22) and (25) for $D_0/(v_0^2 T_{sw}) = 0.005\,74$ and $K = 1.83$.

4. Summary and conclusions

In order to mimic the passive scalar diffusion in a turbulent flow we have considered the diffusion in a homogeneous, isotropic, stationary random velocity field non delta-correlated in time with correlation functions (17) and (18). Using the multiscale technique, we have then studied the non-dimensional diffusion $(D(K) - D_0)/(v_0^2 T_{sw})$ as a function of the Kubo number K in the case of three different energy spectra: the first one proportional to k^{-3}, the second one proportional to $k^{-\frac{5}{3}}$ in the inertial range and the third one proportional to k^2. The first two spectra correspond respectively to the *direct enstrophy cascade regime* and *inverse energy cascade regime* of the 2D turbulence and their power law $k^{-\gamma}$ with $\gamma > 1$ ensures local interactions between the scales in the inertial range. The third one corresponds to the equipartition of kinetic energy among all spatial Fourier modes in three dimensions and, having exponent $\gamma < 1$, it does not correspond to a velocity field dominated by the infrared modes of the spectrum as in the previous cases. In all the cases we have found a linear behaviour for $K \ll 1$ and a power behaviour $K^{-\alpha}$ for finite $K \gg 1$, with $\alpha \simeq 0.165$ in the first two cases and $\alpha \simeq 0.045$ in the third one. The linear behaviour of $D(K)$ is predicted by different analytical approximations whereas for the power behaviour there exists an estimate of the exponent α by Isichenko *et al* [16, 19] for 2D random, incompressible, time-dependent flows with $D_0 = 0$ suggesting the value $\frac{3}{10}$. Ottaviani [18] found the value $\alpha = 0.2 \pm 0.04$ for the random field specified in (14), that is with a unique correlation time and the spectrum proportional to $k^3 e^{-bk^2}$.

The indication arising from our numerical analysis is that the power law behaviour of the diffusivity $D(K)$ for finite $K \gg 1$ is not universal but depends on the spectrum $E(k)$ and the characteristic times at different scales of the considered system. However, the values of the first two exponents α suggest that the power law of the diffusivity could be universal for the velocity field with *local* spectra.

Acknowledgments

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Appendix A

Let us consider a bidimensional random incompressible velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, with correlation tensor

$$\langle v_i(\boldsymbol{x}, t) v_j(\boldsymbol{x}', t')\rangle = C_{ij}(\boldsymbol{x}, \boldsymbol{x}', t, t'). \tag{A1}$$

If the random field is Gaussian, homogeneous, statistically stationary, isotropic, incompressible, zero average, the correlation tensor $C_{ij}(\boldsymbol{r}, \tau)$ can be written as follows [29]:

$$C_{ij}(\boldsymbol{r}, \tau) = \delta_{ij} C_\parallel(r, \tau) + \frac{r_i r_j}{r^2}(C_\parallel(r, \tau) - C_\perp(r, \tau)) \tag{A2}$$

where $C_\parallel = \langle v_\parallel(\boldsymbol{x}, t) v_\parallel(\boldsymbol{x}', t')\rangle$, $C_\perp = \langle v_\perp(\boldsymbol{x}, t) v_\perp(\boldsymbol{x}', t')\rangle$ and $v_\parallel = \boldsymbol{v} \cdot \hat{\boldsymbol{i}}_r$, $v_\perp = \boldsymbol{v} \cdot \hat{\boldsymbol{i}}_\perp$. The incompressibility condition leads to

$$C_\perp = C_\parallel + \frac{r}{2}\partial_r C_\parallel. \tag{A3}$$

For incompressible flow, the Fourier transform $\hat{C}_{jq}(\boldsymbol{k}, \tau)$ of $C_{ij}(\boldsymbol{r}, \tau)$ has the simple expression

$$\hat{C}_{jq}(\boldsymbol{k}, \tau) = \Pi_{jq}(\boldsymbol{k}) f(k, \tau) \tag{A4}$$

where

$$\Pi_{jq}(\boldsymbol{k}) = \delta_{jq} k^2 - k_j k_q \, f(k, \tau) = -\frac{2}{k}\partial_k \hat{C}_\parallel(k, \tau). \tag{A5}$$

The relation between the energy spectrum $E(k)$ of the flow and the function $f(k)$ can be established as follows: by definition the total energy of a fluid is

$$\begin{aligned} E &= \sum_i \langle v_i v_i\rangle(0, 0) = \sum_i C_{ii}(0, 0) \\ &= \sum_i \frac{1}{(2\pi)^2}\int \mathrm{d}\boldsymbol{k} \hat{C}_{ii}(\boldsymbol{k}, 0) \\ &= \sum_i \frac{1}{(2\pi)^2}\int \mathrm{d}\boldsymbol{k} \Pi_{ii}(\boldsymbol{k}) f(k, 0). \end{aligned} \tag{A6}$$

Assuming $f(k, 0) = f(k)$, we derive

$$E = \frac{1}{(2\pi)^2}\int \mathrm{d}\boldsymbol{k}\, k^2 f(k) = \frac{1}{(2\pi)^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}k\, k^3 f(k). \tag{A7}$$

From (A7) the relation between $E(k)$ and $f(k)$ for bidimensional flows is deduced:

$$E(k) = \frac{1}{2\pi} k^3 f(k). \tag{A8}$$

Let us now consider the case of a correlation tensor such that

$$C_{ij}(\boldsymbol{r}, t - t') = R(\boldsymbol{r}) T(|t - t'|) \qquad \text{with} \quad T(|t - t'|) = \frac{D_0}{\tau_0} \mathrm{e}^{-\frac{|t-t'|}{\tau_0}}. \tag{A9}$$

In the Fourier space (A9) becomes

$$\langle \hat{v}_i(\boldsymbol{k}, t) \hat{v}_j(\boldsymbol{k}', t')\rangle = (2\pi)^2 \delta(\boldsymbol{k} + \boldsymbol{k}') \hat{R}_{ij}(\boldsymbol{k}) T(|t - t'|) \tag{A10}$$

where, for the supposed isotropy of the velocity field

$$\hat{R}_{ij}(\boldsymbol{k}) = (k^2 \delta_{ij} - k_i k_j) f(k). \tag{A11}$$

Appendix B

The values of the spectra parameters A_i, B_i, C_i, D_i and G_i depend on k_m^I and k_M^I as follows:

Case 1.

$$A_1 = B_1 {k_m^I}^{-6} \qquad B_1 = v_0^2 \left[\frac{3}{4{k_m^I}^2} - \frac{1}{2{k_M^I}^2} + \frac{1}{{k_M^I}^3} \right]^{-1} \qquad C_1 = B_1 {k_M^I}^{-3} \mathrm{e}^{k_M^I} \tag{B1}$$

$$D_1 = {k_m^I}^3 \qquad G_1 = {k_M^I}^{3/2} \mathrm{e}^{-k_M^I/2} \qquad c_\tau = {B_1}^{-1/2}. \tag{B2}$$

Case 2.

$$A_2 = B_2 {k_m^I}^{-14/3} \qquad B_2 = \frac{8 v_0^2 {k_m^I}^{2/3} {k_M^I}^{5/3}}{14 {k_M^I}^{5/3} + (8 - 12 k_M^I) {k_m^I}^{2/3}} \qquad C_2 = B_2 {k_M^I}^{-5/3} \mathrm{e}^{k_M^I} \tag{B3}$$

$$D_2 = {k_m^I}^{7/3} \qquad G_2 = {k_M^I}^{5/6} \mathrm{e}^{-k_M^I/2} \qquad c_\tau = {B_2}^{-1/2}. \tag{B4}$$

Case 3.

$$A_3 = B_3 {k_m^I}^{-1} \qquad B_3 = \frac{12 v_0^2}{-{k_m^I}^3 + 4{k_M^I}^3 + 12{k_M^I}^2} \qquad C_3 = B_3 {k_M^I}^2 \mathrm{e}^{k_M^I} \tag{B5}$$

$$D_3 = {k_m^I}^{1/2} \qquad G_3 = {k_M^I}^{-1} \mathrm{e}^{-k_M^I/2} \qquad c_\tau = {B_3}^{-1/2}. \tag{B6}$$

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